**Suggested solutions to the IO (BSc) exam on June 16, 2010**
**VERSION: July 1, 2010**

## Question 1

a) **Consider a firm that has a monopoly in a market where demand is given by $q = D(p)$. Let $C(q)$ denote the firm's cost function. Define the Lerner index and then derive the inverse elasticity rule. Explain the intuition behind this rule — how and why does the elasticity matter?**

- The Lerner index (LI) is defined as follows:

$$LI \equiv \frac{p - MC}{p},$$

where MC denotes marginal cost and $p$ is the price. It is a measure of market power, which takes values between zero and one. LI=0 corresponds to no market power at all (the perfect competition outcome) as then price equals marginal cost.

- The inverse elasticity rule says that the value of the Lerner index for a profit maximizing monopolist equals $1/\varepsilon(p^m)$, where $\varepsilon(p^m)$ is the demand elasticity,

$$\varepsilon(p^m) \equiv -\frac{D'(p^m)\, p^m}{D(p^m)},$$

evaluated at the optimal monopoly price, $p^m$. To see this, consider the monopolist's problem:

$$\max_p \{pD(p) - C[D(p)]\}.$$

The solution, $p^m$, must satisfy the following first-order condition:

$$D(p^m) + p^m D'(p^m) = C'[D(p^m)]\, D'(p^m).$$

Dividing through by $D'(p^m)$, using the notation $C'[D(p^m)] = MC$, and rewriting a bit yield

$$p^m - MC = -\frac{D(p^m)}{D'(p^m)} \Leftrightarrow \frac{p^m - MC}{p^m} = -\frac{D(p^m)}{p^m D'(p^m)} = -\frac{1}{\frac{D'(p^m)p^m}{D(p^m)}}$$

or LI=$1/\varepsilon(p^m)$. The inverse elasticity rule thus says that the higher the elasticity, the less market power (as measured by the Lerner index) the monopoly has. The economic reason for this result is simply that a high elasticity means that the consumers are very sensitive to price changes — an increase in the price has a relatively strong negative impact on the amount the consumers are willing to buy. That makes it harder for the firm to (profitably) raise the price significantly above its marginal cost.

b) **State the Coase conjecture. Explain the intuition.**

- The Coase conjecture concerns a situation where a monopoly firm, in each one of many periods, sells a good that is durable. The firm is allowed to choose a new price in each period. The fact that the good is durable means that those costumers who have bought the good will not need to purchase the good in any future period — these customers disappear from the demand. The Coase conjecture (it was later proven to, under certain conditions, hold as a result) states that:
  - When the length between time periods become smaller (or, equivalently, when the consumers' discount factor approaches one), the monopolist's profit converges to the marginal cost — the firm loses all its market power.
- The reason why this happens is that for any given price in a period, the consumers that find it worthwhile to purchase will be those with the highest valuation. That means that in the next period, those high-valuation consumers are not part of demand and therefore the optimal monopoly price must be lower (since demand is lower). In other words, if the monopoly firm cannot precommit to some sequence of prices but is optimizing in each period given the current demand, the price will gradually drop. However, if the consumers understand this they should have an incentive to wait with purchasing until a later period when the price has fallen. The only thing that may stop the consumers from waiting is that they are impatient and prefer immediate consumption to later, all else being equal. But if the length of time between periods is small or if the consumers are not very impatient (which is the condition in the conjecture), then the consumers don't mind waiting until the price has dropped. If so, the firm may be better off lowering the price straight ahead, so that it doesn't have to wait so long for its (perhaps small) profits.

c) **Explain briefly the conjectural-variations approach to modelling an oligopoly.**

- The idea is to assume that (in, say, a duopoly) the firms *believe* (i.e., form a *conjecture*) that a change in one firm's output leads to a change in the rival's output, even though the firms' choices are otherwise modelled as being simultaneous. The degree to which the rival's output changes is captured by a parameter, the conjectural variations parameter. This parameter is typically assumed to be constant (and often also identical across firms). As this parameter takes various values, the outcome of the model (the equilibrium quantities) can be made identical to, for example, the outcome under Cournot or Bertrand competition or the collusive outcome. The approach is therefore used as a reduced-form way of capturing a family of different models with different degrees of competition.

d) **Consider the circular city model (as described by Tirole).**

(i) **Explain in words (you do not need to derive the result formally) how the equilibrium number of entering firms relates to the socially optimal number of entering firms (i.e., the number of firms that minimizes the sum of the firms' entry costs and the consumers' transportation costs). What is the intuition?**

- The socially optimal number of firms takes into account the transportation costs incurred by the consumers and the entry costs incurred by the firms (the amount of trade in the model is held constant by assumption, as we are considering a situation where the market is covered). Having a large number of entering firms saves on the transportation costs whereas having a small number of entrants saves on the entry costs — that's the tradeoff. By writing up an expression for the aggregate costs and then find the number of firms that minimize that expression, one obtains the result that the socially optimal number of firms is exactly one-half of the equilibrium number derived above. That is, at the equilibrium there is too much entry.
- Intuition: A "business-stealing effect" (or "trade-diversion effect" in Tirole):
  - The firms have a private incentive to enter and take sales from the other firms — but this does not add to aggregate welfare as it just amounts to a transfer of profits from one firm to another.

1. (ii) **In the circular city model the locations of the firms are given exogenously. However, in other models they are not. In his discussion, Tirole mentions one force that tends to make firms differentiate, and three forces that oppose product differentiation. Briefly discuss these four forces.**

- The force that makes firms differentiate: By locating far away from a rival you can soften price competition. The forces that make firms want to be close to each other: (1) They want to be where demand is. (2) There may be externalities (a) on the supply side: (fishermen sharing a harbor, firms wanting to be close to a common source of raw material) or (b) on the demand side (firms being close lowers consumers' search costs). (3) There might not be any price competition, for example because the market is regulated.
- See Tirole pp 286-7 for further discussion.

**Question 2**

**Consider two vertically related monopoly firms. The upstream firm produces its good using the cost function $C(q) = \frac{1}{2}q^2$, where $q$ is the quantity produced. It chooses a linear wholesale price, denoted $p_w$. The downstream firm is a retailer and sells the good that the upstream firm produces to the final consumers. The demand of the final consumers is given by $D(p) = 30 - p$, where $p$ is the price chosen by the downstream firm. This firm does not to have any additional costs on top of the costs of purchasing the good from the upstream firm at the wholesale (per unit) price $p_w$. The sequence of events is as follows. First the upstream firm chooses $p_w$; thereafter, knowing $p_w$, the downstream firm chooses $p$.**

a) **Solve for the subgame perfect Nash equilibrium of the model described above. Calculate the equilibrium level of profits of each firm and the equilibrium level of consumer surplus in the market.**

- In order to solve for the subgame perfect equilibrium (or, possibly, equilibria) we can use backward induction. That is, we start with solving the second and last stage of the game, for any given value of $p_w$. Thereafter we plug the resulting expression for $p$ (as a function of $p_w$) into the upstream firm's objective function and then solve for $p$.

- Thus, consider the second stage problem. The downstream firm's profit:

$$\pi_D = (p - p_w)(30 - p).$$

The first-order condition:[1]

$$\frac{\partial \pi_D}{\partial p} = (30 - p) - (p - p_w) = 0 \Rightarrow p^*(p_w) = \boxed{\tfrac{30+p_w}{2}.} \quad (1)$$

- Next we solve the first-stage problem, taking $p^*(p_w)$ into account. The upstream firm's profit:

$$\begin{aligned} \pi_U &= p_w\left[30 - p^*(p_w)\right] - \frac{1}{2}\left[30 - p^*(p_w)\right]^2. \\ &= p_w\left[30 - \frac{30+p_w}{2}\right] - \frac{1}{2}\left[30 - \frac{30+p_w}{2}\right]^2 \\ &= p_w\left[\frac{30-p_w}{2}\right] - \frac{1}{2}\left[\frac{30-p_w}{2}\right]^2. \end{aligned}$$

[1] One can check that the second-order condition for this problem and the one below will be satisfied. One can also verify that the solutions will be interior. A careful student may want to note these things.

The first-order condition:

$$\frac{d\pi_U}{dp_w} = \left[\frac{30-p_w}{2}\right] - p_w\left[\frac{1}{2}\right] + \frac{1}{2}\left[\frac{30-p_w}{2}\right] = 0 \Rightarrow \frac{5}{4}p_w = \frac{90}{4} \Rightarrow p_w = 18.$$

- Plugging $p_w = 18$ into (1) yields $p^*(10) = \frac{30+18}{2} = 24$. So the equilibrium levels of the two prices are

$$\boxed{p_w^* = 18, \qquad p^* = 24.}$$

- The profits levels of the two firms, at the equilibrium, can be calculated as

$$\pi_D^* = (p^* - p_w^*)(30 - p^*) = (24 - 18)(30 - 24) = \boxed{36}$$

(for the downstream firm) and

$$\begin{aligned} \pi_U^* &= p_w\left[\frac{30-p_w}{2}\right] - \frac{1}{2}\left[\frac{30-p_w}{2}\right]^2 = 18\left[\frac{30-18}{2}\right] - \frac{1}{2}\left[\frac{30-18}{2}\right]^2 \\ &= 18 \times 6 - \frac{1}{2} \times 6^2 = 18 \times 5 = \boxed{90} \end{aligned}$$

(for the upstream firm).

- The consumer surplus (CS) is defined (in geometric terms) as the area above the equilibrium price and below the demand curve. Given the assumed linearity of the demand curve, this area has the form of a triangle and the formula for its size is $(q^*)^2/2$, where $q^* = 30 - p^*$ is the quantity bought by the final consumers, given the equilibrium price $p^* = 24$. We thus get

$$CS^* = \frac{(q^*)^2}{2} = \frac{(30-p^*)^2}{2} = \frac{(30-24)^2}{2} = \boxed{18.}$$

b) **Suppose the firms merge. Solve for the retail price that maximizes the profits of the integrated firm. Calculate the equilibrium level of profits of the integrated firm and the equilibrium level of consumer surplus in the market.**

- The profit of the integrated firm equals the sum of the upstream firm's and the downstream firm's profit:

$$\begin{aligned} \pi_I &\equiv \pi_U + \pi_D = p_w[30-p] - \frac{1}{2}[30-p]^2 + (p - p_w)(30 - p) \\ &= p(30-p) - \frac{1}{2}[30-p]^2 . \end{aligned}$$

Thus, the level of the wholesale price $p_w$ is irrelevant, as under integration the good is produced inside the firm instead of being bought from the outside.

- The first-order condition with respect to the only relevant decision variable, $p$, is:

$$\frac{d\pi_I}{dp} = (30 - p) - p + [30 - p] = 0 \Rightarrow 3p = 60 \Rightarrow \boxed{p^{**} = 20.}$$

- The integrated firm's profit at this optimal price is

$$\begin{aligned} \pi_I^{**} &= p^{**}(30 - p^{**}) - \frac{1}{2}[30 - p^{**}]^2 = 20(30 - 20) - \frac{1}{2}[30 - 20]^2 \\ &= 200 - 50 = \boxed{150.} \end{aligned}$$

- The consumer surplus at the optimal price (using the above formula) is

$$CS^{**} = \frac{(q^{**})^2}{2} = \frac{(30 - p^{**})^2}{2} = \frac{(30 - 20)^2}{2} = \boxed{50.}$$

c) **[You are encouraged to attempt parts c) and d) even if you have not been able to answer parts a) and b).] Explain in words what is meant by "double marginalization" and the intuition behind this phenomenon. Relate your answer to the results you have provided under a) and b).**

- "Double marginalization" refers to a situation where two vertically related firms interact, and this interaction leads to a retail price that is too high from all parties point of view (both firms and all consumers). The reason for this is that there is an externality between the two firms: the retailer does not take into account the effect its choice of $p$ has on the manufacturer's profit. Of course, there is an externality also between two horizontally related firms. But then the typical situation is that the goods are *substitutes*: firm 1's demand *drops* if firm 2 lowers its price, yielding an equilibrium price that is too low relative to the joint-profit maximizing price (from the two firm's point of view — the consumers will of course be better off from a lower price). In the vertical story, the input good and the final good are *complements*, so the externality works in the opposite direction. In a horizontal relationship with a demand complementarity, we can again get an equilibrium price that is too high also from the firms' point of view.

- The results under a) and b) illustrate the double marginalization phenomenon.

  - In a), the two firms are separate units and thus they do not internalize the externality that exists between them. The retail price in this case is $p^* = 24$ and the firms' joint profits are $\pi_D^* + \pi_U^* = 36 + 90 = 126$. The consumer surplus is $CS^* = 18$.

– In b), the firms are integrated, so they internalize the externality and can avoid the double marginalization. The retail price is $p^{**} = 20$, which is lower than under non-integration. This lower level of the retail price is indeed making everyone in this economy better off: the firms' joint profit is now $\pi_I^{**} = 150$ (which is higher than under a) and the consumer surplus is now $CS^{**} = 50$ (which also is higher than under a).

d) **Suppose the two firms remain separate as in a). Specify a two-part tariff that — if this is charged by the upstream firm instead of the linear price — gives rise to the same consumer price and level of joint profits (i.e., the sum of upstream and downstream profits) as under integration. Also explain why the two-part tariff achieves that outcome.**

- One two-part tariff that would achieve the outcome under integration is the following:

$$\boxed{(T_w, p_w) = (100, 10),}$$

where $T_w$ is the fixed fee the downstream firm must pay in order to purchase any positive quantity of the upstream firm's good and $p_w$ is the price that in addition must be paid for each unit purchased. The price $p_w = 10$ is chosen so that it equals the upstream firm's marginal cost if producing the socially optimal quantity derived in b). The fixed fee $T_w = 100$ is chosen so that it equals the downstream firm's profits given that is facing this two-part tariff and chooses the retail price optimally. That is, this two-part tariff achieves the desired outcome since it ensures that the downstream firm faces the appropriate marginal cost (which is such that the externality is internalized). In addition, the fixed fee is low enough for the downstream firm to have an incentive to (at least weakly) prefer to purchase from the upstream firm rather than shutting down. We could also have chosen a lower fixed fee, which would have ensured that more of the firms' joint profits were allocated to the downstream firm — such a fixed fee would still have achieved the outcome described in the question, although it wouldn't have maximized the upstream firm's profits.

  – How to find $p_w = 10$: If the downstream firm acts optimally given some two-part tariff $T_w + p_w q$, then its optimal choice of $p$ is given by.$p^*(p_w) = \frac{30+p_w}{2}$ (see (1) above). For the outcome to be the same as under integration, $p_w$ must thus be chosen to ensure that that $\frac{30+p_w}{2} = 20$, or $p_w = 10$.

**Question 3**

**Consider Tirole's version of the Green-Porter model (exactly the same version as we studied in the course). In a market there are two identical firms, firm 1 and firm 2. They produce a homogeneous good and each firm has a constant marginal cost $c \geq 0$. There are infinitely many, discrete time periods $t$ (so $t = 1, 2, 3, \ldots$), and at each $t$ the firms simultaneously choose their respective price, $p_1^t$ and $p_2^t$. The firms' common discount factor is denoted $\delta \in (0, 1)$. As the good is homogeneous, demand is a function of the lowest price, $p^t = \min\{p_1^t, p_2^t\}$. Demand is stochastic: with probability $1 - \alpha$ (where $\alpha \in (0, 1)$), demand in period $t$ is high, $q^t = D(p^t)$ $(> 0)$; and with probability $\alpha$, demand in period $t$ is low — indeed, equal to zero. Demand realizations are independent across time. If the firms charge the same price they share demand equally between themselves. Therefore, firm 1's demand is (the expression for firm 2's demand is analogous):**

$$D_1\left(p_1^t, p_2^t\right) = \begin{cases} D\left(p_1^t\right) & \text{if } p_1^t < p_2^t \text{ and high state} \\ \frac{1}{2} D\left(p_1^t\right) & \text{if } p_1^t = p_2^t \text{ and high state} \\ 0 & \text{if } p_1^t > p_2^t \text{ or low state.} \end{cases}$$

**The firms cannot observe the price charged by the rival firm (not even the prices charged in previous periods). Moreover, the firms cannot observe the state of demand. However, in each period, after having chosen their prices, the firms observe their own demand, although not their rival's demand.**

a) **Let $p^m$ be the high demand monopoly price, i.e., the price that maximizes $(p - c) D(p)$. Consider a trigger strategy (the same one as we studied in the course) where each firm charges $p^m$ until at least one firm makes a zero profit; the occurrence of a zero profit triggers a punishment phase, which lasts for $T$ periods; after the $T$ periods the firms revert to the collusive phase and charge $p^m$ as long as they both make positive profits. Suppose that $\alpha < \frac{1}{2}$. Derive a condition (stated in terms of a critical level of $\delta$) under which a subgame perfect Nash equilibrium in which the firms follow the trigger strategy exists. Interpret your result.**

- In order to derive that condition, first introduce the following notation:
  - Denote by $V^+$ a firm's expected overall equilibrium payoff at the point in time when it is choosing the price and when not being in a punishment phase.
  - Denote by $V^-$ a firm's expected overall equilibrium payoff at the point in time when it is choosing the price and when just having started a punishment phase.

- The trigger strategy as specified in the question does not say explicitly what price level the firms should revert to during the punishment phase. However, the assumption that is natural to make (which also Tirole makes and which we made in the lecture slides) is that the firms set their prices equal to the marginal cost during the punishment phase, thus making a zero profit. (The students may, in principle, make some other assumption but they should specify what that is and then, of course, solve the model correctly given that assumption. However, I cannot see how it will be possible to sustain a Nash equilibrium that is subgame perfect if assuming that the firms revert to some other price.)

- Given that the firms make a zero profit during the punishment phase and the profit $\frac{\Pi^m}{2}$ (half of the optimized monopoly profit) when not in a punishment phase, we have, by definition, these relationships between $V^+$ and $V^-$:
$$V^+ = (1-\alpha)\left[\frac{\Pi^m}{2} + \delta V^+\right] + \alpha\left[0 + \delta V^-\right]$$
and
$$V^- = \overbrace{0 + \delta \times 0 + \delta^2 \times 0 + \cdots + \delta^{T-1} \times 0}^{T\text{-period punishment phase}} + \delta^T V^+ = \delta^T V^+$$
(these expressions should be explained). Solving these two equations for $V^+$ and $V^-$ yields
$$V^+ = \frac{(1-\alpha)\frac{\Pi^m}{2}}{1-(1-\alpha)\delta - \alpha\delta^{T+1}} \qquad \text{and} \qquad V^- = \frac{(1-\alpha)\delta^T\frac{\Pi^m}{2}}{1-(1-\alpha)\delta - \alpha\delta^{T+1}}.$$

- For the trigger strategy to be part of a subgame perfect Nash equilibrium, there are two requirements. First, a firm must not have an incentive to deviate from the strategy when not being in a punishment phase. Second, a firm must not have an incentive to deviate on any occasion during the punishment phase. The latter requirement is clearly satisfied, as the trigger strategy prescribes each firm to set their price equal to the marginal cost. Doing that is optimal if expecting the other firm to do the same (i.e., it's a Nash equilibrium), so that requirement is satisfied. The first requirement can be written as
$$V^+ \geq \underbrace{(1-\alpha)\left[\Pi^m + \delta V^-\right]}_{\text{dev yields large profit}} + \underbrace{\alpha\left[0 + \delta V^-\right]}_{\text{dev won't matter}}. \qquad \text{(Nash)}$$
Using our expressions for $V^+$ and $V^-$ above, the above condition can be rewritten (after some algebra, which should be shown) as
$$1 \leq 2(1-\alpha)\delta + (2\alpha - 1)\delta^{T+1}. \qquad \text{(Nash)}$$

- The right-hand side of this expression is increasing in $T$ if $\alpha < \frac{1}{2}$ (which we are supposed to assume, according to the question). That is, the longer the punishment period, the easier it is to satisfy the inequality and the less attractive it is to deviate from the trigger strategy. The fact that the right-hand side is increasing in $T$ means that we can investigate the circumstances under which cooperation is possible for at least some punishment period $T$ by looking at the extreme case where $T = \infty$. Taking the limit $T \to \infty$ of the right-hand side of (Nash) yields (since $\delta \in (0, 1)$) the following version of the inequality:

  $$1 \leq 2 (1 - \alpha) \delta.$$

  By solving for $\delta$, we obtain

  $$\delta \geq \frac{1}{2 (1 - \alpha)}.$$

  This condition says that the discount factor $\delta$ must be large enough for cooperation to be possible. Intuitively, the firms must care sufficiently much about future profits, for the bad thing with deviating is that then the firm loses high profits in the future and the good thing with deviating is that this makes the profits in the current period exceptionally large — hence the firm mustn't discount the future too much or otherwise it will have an incentive to deviate. We also see that the requirement on $\delta$ becomes more stringent, the larger $\alpha$ is. Intuitively, a high $\alpha$ means that a low state and hence the inference problem occurs more often — it will happen more frequently that the firm cannot distinguish between a deviation of the rival and a low demand. That is why collusion is harder for large values of $\alpha$.

b) **[You are encouraged to attempt part b) even if you have not been able to answer part a).] In the course we studied two theories that give rise to specific predictions about the relationship between the business cycle and the likelihood of a price war. The Green-Porter model was one of these. Give a brief account (in words) of the other theory and explain how and why the predictions of the two theories differ from each other.**

- The other theory is the one proposed by Rotemberg and Saloner. Rotemberg and Saloner's model predicts price wars during booms, meaning counter-cyclical prices (although with a caveat, explained below). In contrast, Green and Porter's model predicts price wars during recessions, meaning pro-cyclical prices.

- Both models assume an infinitely repeated duopoly game with price competition. Both models also assume that demand fluctuates. However, it is only in the Green-Porter model that the firms face uncertainty about the current-period demand (as well as about the rival's price, making inference about demand hard). In both models the demand realizations are independent over time.

- Thus, in Rotemberg-Saloner the demand state — whether it's high or low — is common knowledge when the price is set; however future demand realizations, which are independent from the current one, are not known. This creates a stronger incentive to deviate from a cooperative equilibrium when demand is high. This is because a current high demand and the possibility of a lower demand tomorrow makes the "one-period temptation" large relative to the "long-term reward of not deviating" — collusion is hard when demand is unusually high. Rotemberg and Saloner interpret this as price war during booms — there is less collusion in good times. However, in this "price war" the price level may actually be higher during the price war (i.e., during the boom) than otherwise — we may have $p_H^* > p_L^*$. Although the price during a boom is low relative to the high level of demand, it may be high relative to the low-demand price. This is the caveat referred to above.

- In the Green-Porter model that we solve for under a) the firms can't distinguish a price cut of the rival from a low demand state. However, to sustain a collusive equilibrium we must have *some* punishment after a low demand — otherwise there would be an incentive to deviate. Therefore collusion must break down at least temporarily when demand is low (but with a finite punishment phase, collusion can start again in a later period). Thus the price wars (in the sense of some periods with marginal cost pricing) will break out every time demand is low. The reason for this is that a deviation by the rival amounts to that firm undercutting the first firm — hence all the consumers will go the rival firm. Hence the first firm, which cannot observe directly what price the rival firm has charged, only knows that it did not get any costumers. It will not be able to tell whether this was because the rival deviated or because demand in low in this period. This creates an inference problem which cannot be solved, and to sustain cooperation in high-demand states the collusion must break down whenever a low-demand state occurs.

END OF EXAM